J. Fluid Mech. (2007), *vol.* 584, *pp.* 185–202.
doi:10.1017/S0022112007006386 Printed in the United Kingdom

Two-layer quasi-geostrophic singular vortices embedded in a regular flow. Part 1. Invariants of motion and stability of vortex pairs

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(Received 5 October 2006 and in revised form 6 March 2007)

The concept of a quasi-geostrophic singular vortex is extended to several types of two-layer model: a rigid-lid two-layer, a free-surface two-layer and a $2\frac{1}{2}$-layer model with two active and one passive layer. Generally, a singular vortex differs from a conventional point vortex in that the intrinsic vorticity of a singular vortex, in addition to delta-function, contains an exponentially decaying term. The theory developed herein occupies an intermediate position between discrete and fully continuous multilayer models, since the regular flow and its interaction with the singular vortices are also taken into account. A system of equations describing the joint evolution of the vortices and the regular field is presented, and integrals expressing the conservation of enstrophy, energy, momentum and mass are derived. Using these integrals, the initial phases of evolution of an individual singular vortex confined to one layer and of a coaxial pair of vortices positioned in different layers of a two-layer fluid on a beta-plane are described. A valuable application of the conservation integrals is related to the stability analysis of point-vortex pairs within the $1\frac{1}{2}$-layer model, $2\frac{1}{2}$-layer model, and free-surface two-layer model on the f-plane. Such vortex pairs are shown to be nonlinearly stable with respect to any small perturbation provided its regular-flow energy and enstrophy are finite.

1. Introduction

Intense localized synoptic (mesoscale) eddies in the atmosphere and oceans are remarkably durable and significantly contribute to the transport of kinetic energy, heat, momentum and material (both chemical and biological). As a rule, localized vortices are strongly nonlinear, their dynamics being affected by a number of physical factors, such as the β-effect (the planetary vorticity gradient), background large-scale flows, density stratification and topography. Generally, complete incorporation of all these effects into one model is problematic. Therefore, simplified models that correctly reproduce the underlying physics are employed. One such approach is based on approximation of distributed eddies by systems of singular vortices.

The form of singular vortices is not universal and can be chosen based on physical considerations. On the f-plane, where the β-effect is absent (constant rotation), a natural choice is to use conventional point vortices for singular vortices. Within the limits of this approach, the potential vorticity (PV) of a point vortex is concentrated

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in a point and is proportional to Dirac's delta-function. Any f-plane vortical flow can be approximated by a sufficiently large set of point vortices, whose dynamics are described by a system of ordinary differential equations. In a free-surface f-plane model, the combined effect of rotation and stratification manifests itself in that the peripheral velocity field of a discrete vortex decays exponentially (see Obukhov 1949; Morikawa 1960), rather than algebraically, as in classical two-dimensional fluid dynamics. Systems of point vortices in a constantly rotating stratified fluid have been considered in various physical contexts (Gryanik 1983; Gryanik & Tevs 1989, 1991, 1997; Hogg & Stommel 1985*a*, *b*; Pedlosky 1985; Young 1985; Griffiths & Hopfinger 1986; Legg & Marshall 1993; Sokolovskiy & Verron 2000, 2002, 2004; Kizner 2006; reviewed by Gryanik, Sokolovskiy & Verron 2006).

Consideration of the β-effect complicates assessment of the dynamics of singular vortices, because a non-zonal displacement of a fluid parcel is always associated with a change in its intrinsic vorticity, which is defined here as the quasi-geostrophic PV minus planetary vorticity. A concept of barotropic modulated β-plane point vortices was suggested by Zabusky & McWilliams (1982). As with f-plane point vortices, where PV and intrinsic vorticity coincide, the motion of an ensemble of modulated β-plane point vortices is described by ordinary differential equations; however, the circulation (or strength) of any vortex is assumed to change upon non-zonal displacement of the vortex. Using this approach, qualitative analysis of the dynamics of barotropic vortices was achieved (Zabusky & McWilliams 1982; Kono & Horton 1991; Hobson 1991; Velasco Fuentes & van Heijst 1994, 1995). A two-layer version of the β-modulation was used by Kizner (2006) to examine the involvement of the β-effect in transitions of baroclinic modon-like vortical configurations. However, β-modulation model cannot be formally derived from the equations of PV conservation, but rather is an approximate approach intended to imitate the β-effect in vortical systems (Reznik 1992).

As distinct from conventional point vortices, a rigorous approach to the β-plane dynamics involves a more general class of singular vortices. As with a free-surface f-plane point vortex, the velocity field in a singular vortex decays exponentially at infinity. However, the intrinsic vorticity of a singular vortex, unlike that of conventional point vortices, contains an exponentially decaying component, in addition to the delta-function component. On the β-plane, vortices can form ensembles that travel steadily in the zonal direction without generating any regular velocity field besides the singular velocity field induced by the vortices themselves (Reznik 1986, 1992; Gryanik 1986, 1988; Flierl 1987; Gryanik, Borth & Olbers 2004). This scenario is possible if the amplitudes and coordinates of the vortices are appropriately fitted.

The situation differs significantly when an additional regular flow is present. On the f-plane, the regular flow will exist only if it is present in the initial state. In contrast, on the β-plane, a regular-flow component will develop because of a non-stationary motion of a singular-vortex ensemble, even if the initial state is free of a regular flow. For instance, an individual singular vortex on the β-plane cannot be stationary. Consequently, the motion of such a vortex will induce development of a regular flow. Obviously, the regular component of a mixed singular–regular flow interacts with the singular vortices, resulting in both the regular and singular fields undergoing changes. Evolution of a combined singular–regular system is described by a set of interrelated equations that govern the regular flow component and the drift of singular vortices. In the barotropic case, such a theory was developed and used to examine the evolution of an intense singular vortex on the β-plane (Reznik 1992).

In this paper, the theory of singular vortices is extended to two-layer fluids. Such a task implies the derivation of basic differential equations and integral invariants of the two-layer mixed singular–regular dynamics. These invariants are used to tackle the problem of stability of vortical pairs. In the absence of regular flow, any pair of point vortices on the *f*-plane, whether barotropic or baroclinic, is stable in the sense that a perturbation of their initial coordinates will not grow with time. However, the question as to how regular-flow perturbations might affect a singular-vortex pair has not been considered until now. We offer an answer to this question by showing that a singular-vortex pair is nonlinearly stable with respect to any small perturbation whose regular-flow energy and enstrophy are finite.

Equations that govern the cooperative evolution of singular vortices and a regular background flow are presented in § 2. Invariants of motion of such a system and some straightforward consequences drawn from these conservation laws are considered in § 3. Stability of *f*-plane vortex pairs is examined in § 4. The main results of the paper are summarized in § 5.

2. Basic equations

2.1. *Conservation of potential vorticity in layered models*

We consider a hierarchy of models, from the $1\frac{1}{2}$-layer to the $2\frac{1}{2}$-layer model, based on the following pair of conventional equations of conservation of quasi-geostrophic potential vorticity:

$$\frac{\partial \Pi_i}{\partial t} + J(\psi_i, \Pi_i) = 0, \quad \Pi = q_i + \beta y, \quad i = 1, 2, \tag{2.1a}$$

Subscripts 1 and 2 are indices of the first (upper) and second (lower) layers; ψ_i, Π_i, and q_i are the streamfunction, PV, and intrinsic vorticity in layer i, respectively; $J(,)$ is the Jacobian operator. The intrinsic vorticities are given by the following equations:

$$q_1 = \nabla^2\psi_1 + \Lambda_1(s_1\psi_2 - \psi_1), \quad q_2 = \nabla^2\psi_2 + \Lambda_2(s_2\psi_1 - \psi_2). \tag{2.1b, c}$$

The constants $\Lambda_1, \Lambda_2, s_1$ and s_2 are defined below using the following conventional notations: f_0 is the reference value of the Coriolis parameter and β its northward gradient; g' is the reduced gravity; and H_i and ρ_i are the depth and fluid density of layer i.

The term '$n\frac{1}{2}$-layer model' is commonly used to refer to an $(n + 1)$-layer fluid, where the upper n layers are assumed to be active and the lower layer, $n + 1$, to be motionless. Such a situation arises when the motion in the lower layer can be ignored owing to the great (infinite) thickness of this layer. In the $2\frac{1}{2}$-layer model (two active and one passive layer):

$$\Lambda_1 = \frac{f_0^2}{g_1' H_1}, \quad \Lambda_2 = \frac{f_0^2}{g_2' H_2}\frac{\rho_3 - \rho_1}{\rho_2 - \rho_1}, \quad s_1 = \frac{\rho_2}{\rho_1}, \quad s_2 = \frac{\rho_1(\rho_3 - \rho_2)}{\rho_2(\rho_3 - \rho_1)}, \tag{2.2}$$

where

$$g_1' = g\frac{\rho_2 - \rho_1}{\rho_1}, \quad g_2' = g\frac{\rho_3 - \rho_2}{\rho_2}, \quad \rho_3 > \rho_2 > \rho_1. \tag{2.3}$$

In the case of a two-layer fluid with a free upper surface

$$\Lambda_1 = \frac{f_0^2}{g' H_1}, \quad \Lambda_2 = \frac{f_0^2}{g' H_2}, \quad s_1 = 1, \quad s_2 = \frac{\rho_1}{\rho_2}, \quad g' = g\frac{\rho_2 - \rho_1}{\rho_2}. \tag{2.4}$$

When a two-layer flow under the rigid-lid condition is considered

$$\Lambda_1 = \frac{f_0^2}{g'H_1}, \quad \Lambda_2 = \frac{f_0^2}{g'H_2}, \quad s_1 = s_2 = 1, \quad g' = g\frac{\rho_2 - \rho_1}{\rho_2}. \tag{2.5}$$

In a $1\frac{1}{2}$-layer model (one active layer and one infinitely deep passive layer or one layer with a free upper surface), equations (2.1) describe the motion in only one layer 1, so the definitions become:

$$\Lambda_1 = \frac{f_0^2}{g'H_1}, \quad \Lambda_2 = 0, \quad \psi_2 \equiv 0, \quad s_1 = 1. \tag{2.6}$$

2.2. *Point vortices*

The invariants of motion and the stability analysis of vortex pairs presented in §§ 3 and 4 depend strongly on the form of singular vortices assumed. In principle, there are different ways, all based upon physical considerations, of determining the form of a singular vortex. If the β-effect is absent, i.e. $\beta = 0$ in (2.1), the best choice is conventional point vortices confined to either the upper or lower layer. An upper-layer point vortex of unit circulation is determined by

$$\left.\begin{aligned} q_{1,s}^u &= \nabla^2\psi_{1,s}^u + \Lambda_1\left(s_1\psi_{2,s}^u - \psi_{1,s}^u\right) = \delta(x)\delta(y),\\ q_{2,s}^u &= \nabla^2\psi_{2,s}^u + \Lambda_2\left(s_2\psi_{1,s}^u - \psi_{2,s}^u\right) = 0. \end{aligned}\right\} \tag{2.7a}$$

Here in after, $\delta(z)$ is Dirac's delta-function, variables marked by subscript s are associated with singular vortices, superscript u indicates that the corresponding fields are induced by the singular vortex located in the upper layer. Note that a singular vortex confined to the upper layer induces flows in both layers. The equations determining a lower-layer unit point vortex are:

$$\left.\begin{aligned} q_{1,s}^l &= \nabla^2\psi_{1,s}^l + \Lambda_1\left(s_1\psi_{2,s}^l - \psi_{1,s}^l\right) = 0,\\ q_{2,s}^l &= \nabla^2\psi_{2,s}^l + \Lambda_2\left(s_2\psi_{1,s}^l - \psi_{2,s}^l\right) = \delta(x)\delta(y), \end{aligned}\right\} \tag{2.7b}$$

where the superscript l indicates the lower location of the singular vortex.

In the quest for the streamfunctions $\psi_{i,s}^u$ and $\psi_{i,s}^l$ (where $i = 1, 2$), we introduce the normal-mode variables

$$\psi^{\pm} = \psi_1 + \alpha^{\pm}\psi_2, \quad q^{\pm} = q_1 + \alpha^{\pm}q_2, \tag{2.8}$$

where, for simplicity, subscript s and superscripts u and l are omitted, and the coefficients $\alpha^{\pm}$ are defined as

$$\alpha^{(\pm)} = \frac{1}{2\Lambda_2 s_2}[(\Lambda_1 - \Lambda_2) \pm \sqrt{(\Lambda_1 - \Lambda_2)^2 + 4\Lambda_1\Lambda_2 s_1 s_2}]. \tag{2.9}$$

When (2.8) and (2.9) are substituted into (2.1b, c), the normal-mode variables (2.8) decouple. Solving equations for the normal-mode variables and reverting to the layer streamfunctions yields:

$$\psi_{1,s}^u = \psi_{1,s}^u(r) = \frac{1}{2\pi(\alpha^+ - \alpha^-)}(\alpha^- K_0(p_- r) - \alpha^+ K_0(p_+ r)), \tag{2.10a}$$

$$\psi_{2,s}^u = \psi_{2,s}^u(r) = \frac{1}{2\pi(\alpha^+ - \alpha^-)}\left(K_0(p_+ r) - K_0(p_- r)\right), \tag{2.10b}$$

$$\psi^l_{1,s} = \psi^l_{1,s}(r) = \frac{\alpha^-\alpha^+}{2\pi(\alpha^+ - \alpha^-)}(K_0(p_-r) - K_0(p_+r)), \tag{2.11a}$$

$$\psi^l_{2,s} = \psi^l_{2,s}(r) = \frac{1}{2\pi(\alpha^+ - \alpha^-)}(\alpha^- K_0(p_+r) - \alpha^+ K_0(p_-r)). \tag{2.11b}$$

From here on $r = \sqrt{x^2 + y^2}$ is the polar radius, and $K_m(z)$ and $I_m(z)$ are the modified m-order Bessel functions of the argument z. At this stage, when conventional point vortices are considered, parameter $p_\pm$ in (2.10) and (2.11) is defined as

$$p^2_\pm = d_\pm, \tag{2.12a}$$

where

$$d_\pm = \tfrac{1}{2}[(\Lambda_1 + \Lambda_2) \pm \sqrt{(\Lambda_1 - \Lambda_2)^2 + 4\Lambda_1\Lambda_2 s_1 s_2}]. \tag{2.12b}$$

However, when dealing with singular vortices on the β-plane and using (2.10) and (2.11), below, $p_\pm$ will be defined differently.

In a two-layer model with a free surface and in a $2\frac{1}{2}$-layer model,

$$s_1 s_2 < 1, \tag{2.13}$$

(see (2.2), (2.4)). Therefore, in these cases

$$d_\pm > 0, \quad p^2_\pm > 0. \tag{2.14}$$

In a two-layer model under the rigid-lid condition, $d_- = p^2_- = 0$. Accordingly, the function $K_0(p_-r)$ in (2.10), (2.11) should be replaced with $-\ln r$. Note that the streamfunction $\psi^u_{1,s}$ (or $\psi^l_{2,s}$) has a logarithmic singularity at $r = 0$, whereas the streamfunction $\psi^u_{2,s}$ (or $\psi^l_{1,s}$) is regular throughout the (x, y)-plane and represents the motion induced in the lower (upper) layer by the singular vortex confined to the upper (lower) layer.

In the absence of a background regular flow, the evolution of a system of point vortices of the type (2.7) is governed by ordinary differential equations. Generally, if $\beta \neq 0$, the flow induced by the singular vortices leads to redistribution of the background PV owing to non-zonal displacements of elements of ambient fluid. This redistribution causes generation of a regular velocity field, which is in addition to the velocity field due to the vortices. In such cases, the evolution of an ensemble of singular vortices cannot be reduced to a system of ordinary differential equations, since the regular flow affects the motion of the singular vortices. Thus, the system of coupled differential equations, which describes the combined evolution of the regular flow and singular vortices, should be considered.

2.3. *Two-layer modons and singular vortices*

In order to properly define the form of a singular vortex (for $\beta \neq 0$), first we consider localized steadily translating vortical solutions to the system of equations (2.1), referring to such solutions as modons. Because of the steady translation, the streamfunctions ψ_1 and ψ_2 of a modon can be considered as depending only on the arguments $x - Ut$ and y:

$$\psi_i = \psi_i(x - Ut, y), \quad i = 1, 2, \tag{2.15}$$

where U is the constant translation speed of the modon (the translation can only be zonal). Therefore, in the co-moving frame of reference, (2.1a) can be rewritten as:

$$J\left(\psi_i + Uy, q_i - \frac{\beta}{U}\psi_i\right) = 0, \quad i = 1, 2. \tag{2.16}$$

In each layer, a bounded interior domain D_i and an infinite exterior domain exist, filled with closed and open streamlines (contours of the co-moving streamfunction $\psi_i + Uy$), respectively (see Flierl *et al.* 1980). In the exterior domain, (2.16) reduces to

$$q_i - \frac{\beta}{U}\psi_i = 0, \quad i = 1, 2. \tag{2.17a}$$

In the interior domain, D_i, (2.16) is equivalent to

$$q_i - \frac{\beta}{U}\psi_i = Z_i(\psi_i + Uy), \quad i = 1, 2, \tag{2.17b}$$

where Z_i is a differentiable function. Any solution to (2.17) that decays sufficiently fast at infinity can be presented implicitly as:

$$\psi_1 = \int_{D_1} Z_1[\psi_1(\boldsymbol{r}') + Uy']\psi_{1,s}^u(|\boldsymbol{r} - \boldsymbol{r}'|)\,\mathrm{d}\boldsymbol{r}' + \int_{D_2} Z_2[\psi_2(\boldsymbol{r}') + Uy']\psi_{1,s}^l(|\boldsymbol{r} - \boldsymbol{r}'|)\,\mathrm{d}\boldsymbol{r}', \tag{2.18a}$$

$$\psi_2 = \int_{D_1} Z_1[\psi_1(\boldsymbol{r}') + Uy']\psi_{2,s}^u(|\boldsymbol{r} - \boldsymbol{r}'|)\,\mathrm{d}\boldsymbol{r}' + \int_{D_2} Z_2[\psi_2(\boldsymbol{r}') + Uy']\psi_{2,s}^l(|\boldsymbol{r} - \boldsymbol{r}'|)\,\mathrm{d}\boldsymbol{r}'. \tag{2.18b}$$

The functions $\psi_{1,s}^u$, $\psi_{2,s}^u$, $\psi_{1,s}^l$ and $\psi_{2,s}^l$ in (2.18) are given by (2.10) and (2.11) with $p_\pm^2 = \beta/U + d_\pm$, where $d_\pm$ is determined by (2.12b). The equations these functions obey are:

$$q_{1,s}^u - p^2\psi_{1,s}^u = \delta(x)\delta(y), \quad q_{2,s}^u - p^2\psi_{2,s}^u = 0, \tag{2.19a}$$

$$q_{1,s}^l - p^2\psi_{1,s}^l = 0, \quad q_{2,s}^l - p^2\psi_{2,s}^l = \delta(x)\delta(y), \tag{2.19b}$$

where $p^2 = \beta/U$. Generally, parameter p^2 can be of any sign. However, for functions $\psi_{1,s}^u$, $\psi_{2,s}^u$, $\psi_{1,s}^l$ and $\psi_{2,s}^l$ and, therefore, for the solution to (2.17) given by (2.18) to decay exponentially with $r \to \infty$, parameter p^2 must satisfy the condition

$$p_\pm^2 = p^2 + d_\pm \geqslant 0. \tag{2.19c}$$

The derivations above were carried out to demonstrate that any two-layer modon can be interpreted as a superposition of a continuum of singular vortices that each have the form (2.19) (within an amplitude factor) and fill the finite interior domains D_i. Unlike the vortices determined by (2.7), those given by (2.19) are not conventional point vortices. This is because the corresponding intrinsic vorticities in the layers, besides Dirac's delta-functions, contain additional terms that decay exponentially at $r \to \infty$ and have a logarithmic singularity at the origin.

In what follows, as the elements that constitute the singular part of a mixed singular–regular solution to (2.1), we use the singular vortices (2.19) of which any modon consists. Several vortices with specially fitted amplitudes and positions can form a discrete modon (see Reznik 1992; Gryanik 1988) or other steadily translated structures (Gryanik *et al.* 2004). However, if the amplitudes and coordinates of vortices are arbitrary, such an ensemble is not necessarily stationary, and p^2 in (2.19) is not related to the parameter β or to any prescribed translation speed U. In other words, p^2 can also be set arbitrarily, with the only restriction imposed on p^2 being condition (2.19*c*), which assures exponential decay of the streamfunctions of the vortex at infinity.

2.4. *Non-stationary systems of singular vortices*

Let that streamfunction ψ_i in each layer be a superposition of the streamfunctions $\psi_{i,r}$ and $\psi_{i,s}$ of the regular and singular flows, respectively:

$$\psi_i = \psi_{i,r} + \psi_{i,s}, \quad i = 1, 2. \tag{20}$$

We assume the singular part of the flow to consist of N_1 upper-layer and N_2 lower-layer singular vortices of the type (2.19):

$$\psi_{1,s} = \sum_{n_1}^{N_1} A_{1,n_1} \psi^u_{1,s}(|\boldsymbol{r} - \boldsymbol{r}_{n_1}|) + \sum_{n_2}^{N_2} A_{2,n_2} \psi^l_{1,s}(|\boldsymbol{r} - \boldsymbol{r}_{n_2}|), \tag{2.21a}$$

$$\psi_{2,s} = \sum_{n_1}^{N_1} A_{1,n_1} \psi^u_{2,s}(|\boldsymbol{r} - \boldsymbol{r}_{n_1}|) + \sum_{n_2}^{N_2} A_{2,n_2} \psi^l_{2,s}(|\boldsymbol{r} - \boldsymbol{r}_{n_2}|), \tag{2.21b}$$

where A_{i,n_i} is the amplitude of vortex n_i in layer i, and $\boldsymbol{r} = \boldsymbol{r}_{n_i}(t)$ is its trajectory, the summation being over $n_1 = 1, \ldots, N_1$ and $n_2 = 1, \ldots, N_2$. The amplitudes of the two-layer singular vortices in (2.21) are constants of motion, as in the case with barotropic vortices (Reznik 1992).

The single-layer derivations of Reznik (1992) can now be re-examined in the framework of a layered model. By substituting (2.20) and (2.21) into (2.1*a*), and setting to zero separately the regular and singular parts, we obtain:

$$\frac{\partial}{\partial t}(q_{i,r} + p^2\psi_{i,s} + \beta y) + J(\psi_{i,r} + \psi_{i,s}, q_{i,r} + p^2\psi_{i,s} + \beta y) = 0, \tag{2.22a}$$

$$\dot{x}_{m_i} = -\left.\frac{\partial\left(\psi_{i,r} + \psi^{m_i}_{i,s}\right)}{\partial y}\right|_{\boldsymbol{r}=\boldsymbol{r}_{m_i}}, \quad \dot{y}_{m_i} = \left.\frac{\partial\left(\psi_{i,r} + \psi^{m_i}_{i,s}\right)}{\partial x}\right|_{\boldsymbol{r}=\boldsymbol{r}_{m_i}}, \quad i = 1, 2. \tag{2.22b, c}$$

The parameter p^2 is a prescribed positive constant; the regular-component intrinsic vorticity is

$$q_{1,r} = \nabla^2\psi_{1,r} + \Lambda_1(s_1\psi_{2,r} - \psi_{1,r}), \quad q_{2,r} = \nabla^2\psi_{2,r} + \Lambda_2(s_2\psi_{1,r} - \psi_{2,r}); \tag{2.23a, b}$$

whereas

$$\psi^{m_1}_{1,s} = \sum_{n_1 \neq m_1}^{N_1} A_{1,n_1} \psi^u_{1,s}(|\boldsymbol{r} - \boldsymbol{r}_{n_1}|) + \sum_{n_2}^{N_2} A_{2,n_2} \psi^l_{1,s}(|\boldsymbol{r} - \boldsymbol{r}_{n_2}|), \tag{2.24a}$$

$$\psi^{m_2}_{2,s} = \sum_{n_1}^{N_1} A_{1,n_1} \psi^u_{2,s}(|\boldsymbol{r} - \boldsymbol{r}_{n_1}|) + \sum_{n_2 \neq m_2}^{N_2} A_{2,n_2} \psi^l_{2,s}(|\boldsymbol{r} - \boldsymbol{r}_{n_2}|), \tag{2.24b}$$

are the upper- and lower-layer singular streamfunctions with vortices m_1 and m_2 excluded from summation.

Equation (2.22*a*) contains singular factors at regular functions, and we assume that for all n_i, the function $\psi_{i,r}$ is infinitely differentiable at $\boldsymbol{r} \neq \boldsymbol{r}_{n_i}$ and doubly differentiable at $\boldsymbol{r} = \boldsymbol{r}_{n_i}$. The regular vorticity $q_{i,r}$ is continuous throughout the (x, y)-plane, but generally $\partial q_{i,r}/\partial t$ and $\nabla q_{i,r}$ have singularities at $\boldsymbol{r} = \boldsymbol{r}_{n_i}$, and the singularities in (2.22*a*) must mutually cancel out. The physical meaning of each of equations (2.22*a*–2.22*c*) is the same as in the one-layer case. Equations (2.22*b*) and (2.22*c*) imply that the motion of a singular vortex is induced by other singular vortices, and by the regular-flow component as well. The most complex equation, (2.22*a*), describes the evolution of the regular-flow streamfunction $\psi_{i,r}$, and thus the quantity $q_{i,r} + p^2\psi_{i,s} + \beta y$ (which can be referred to as the regular PV) is conserved in each fluid element distinct from the singular vortices. This conservation implies that the regular intrinsic vorticity $q_{i,r}$ in the element depends not only on the meridional position y of the element, but also on the disposition of the singular vortices.

When $\beta = 0$ and $p = 0$, the singular vortices are the conventional point vortices, in which case, the regular PV coincides with the regular intrinsic vorticity $q_{i,r}$. If the regular field $\psi_{i,r}$ is zero, (2.22*a*) is satisfied, and (2.22*b*) and (2.22*c*) reduce to the familiar system of ordinary differential equations that describes the motion of an ensemble of interacting point vortices (Gryanik 1983). However, if $\psi_{i,r} \neq 0$ at some moment, then $\psi_{i,r}$ will remain non-zero since, at $p = 0$, regular PV is conserved in any fluid element distinct from the point vortices themselves. When $\beta = 0$ and $p \neq 0$, as occurs in truly singular *f*-plane vortices (2.19), the regular PV, $q_{i,r} + p^2\psi_{i,s}$, of a fluid element depends on the position of the element relative to the singular vortices. An ensemble of such singular vortices ($N_1 + N_2 > 1$ in (2.21)) generates a regular component, $\psi_{i,r}$.

3. Invariants of motion

3.1. *Enstrophy integral*

The derivation of the enstrophy integral of the motion of a mixed singular–regular vortical system is similar to that carried out by Reznik (1992). In the absence of the singular component, this invariant converts into the conventional enstrophy integral, and thus the term 'enstrophy integral' is used. First, the two sides of (2.22*a*) are multiplied by $q_{1,r} - p^2\psi_{1,r}$ at $i = 1$, and by $(\Lambda_1 s_1/\Lambda_2 s_2)(q_{2,r} - p^2\psi_{2,r})$ at $i = 2$, integrated over the (x, y)-plane, and the results summed up. Next, the resulting equation is transformed with the use of equations (2.22*b, c*), (2.12), (2.19) and the formula

$$\int_R (\nabla^2 F - p^2 F) K_0(pr)\,\mathrm{d}x\,\mathrm{d}y = -2\pi F(0,0), \tag{3.1}$$

which is valid for any regular function $F(x, y)$. After some algebra, the following conservation law is obtained:

$$S_r + p^2 K_S - \beta\left(\sum_{m_1} A_{1,m_1} y_{m_1} + \frac{\Lambda_1 s_1}{\Lambda_2 s_2}\sum_{m_2} A_{2,m_2} y_{m_2}\right) = \text{const.} \tag{3.2}$$

In (3.2), the functional K_S is:

$$\begin{aligned} K_S = -\tfrac{1}{2}\Bigg\{ &\sum_{m_1 \neq n_1} A_{1,m_1} A_{1,n_1} \psi^u_{1,s}\left(r_{m_1,n_1}\right) + \frac{\Lambda_1 s_1}{\Lambda_2 s_2}\sum_{m_2 \neq n_2} A_{2,m_2} A_{2,n_2} \psi^l_{2,s}\left(r_{m_2,n_2}\right) \\ &+ 2\frac{\Lambda_1 s_1}{\Lambda_2 s_2}\sum_{m_1,m_2} A_{1,m_1} A_{2,m_2} \psi^u_{2,s}\left(r_{m_1,m_2}\right)\Bigg\}, \end{aligned} \tag{3.3}$$

where $r_{m_i,n_i} = |\boldsymbol{r}_{m_i} - \boldsymbol{r}_{n_i}|$ is the distance between vortices m_i and n_i. Using (2.10) and (2.11), when $\beta = p = 0$, (3.3) reduces to the Kirchoff function for a two-layer ensemble of conventional f-plane point vortices.

Because of inequality (2.13), the functional S_r,

$$S_r = \tfrac{1}{2}\int \left\{ \left[q_{1,r}^2 + p^2(\nabla\psi_{1,r})^2\right] + \frac{\Lambda_1 s_1}{\Lambda_2 s_2}\left[q_{2,r}^2 + p^2(\nabla\psi_{2,r})^2\right] + p^2\Lambda_1\left(\psi_{1,r}^2 + \frac{s_1}{s_2}\psi_{2,r}^2 - 2s_1\psi_{1,r}\psi_{2,r}\right)\right\} \mathrm{d}x\,\mathrm{d}y, \quad (3.4)$$

is positive-definite. In the special case of $\beta = p = 0$, it reduces to the enstrophy due to the regular component. According to (3.2) and (3.3), S_r changes because of changes in the mutual distances between the singular vortices, and because of non-zonal displacements of the vortices (the second and the third terms on the left-hand side of (3.2), respectively).

3.2. *Energy integral*

Another invariant of motion of a mixed singular–regular vortical system, which was not previously considered, can be termed the energy integral, because it converts into a conventional energy integral in the absence of the singular component. This invariant is derived by multiplying the two sides of (2.22*a*) by $\psi_{1,r} + \psi_{1,s}$ at $i = 1$ and by $(\Lambda_1 s_1/\Lambda_2 s_2)(\psi_{2,r} + \psi_{2,s})$ at $i = 2$, integrating them over the (x, y)-plane, and summing up the results. Using (3.1) and (2.22*b*, *c*), the following conservation law is obtained:

$$E_r - p^2 E_{s,r} + K_S - \sum_{m_1} A_{1,m_1}\, \psi_{1,r}|_{\boldsymbol{r}=\boldsymbol{r}_{m_1}} - \frac{\Lambda_1 s_1}{\Lambda_2 s_2}\sum_{m_2} A_{2,m_2}\, \psi_{2,r}|_{\boldsymbol{r}=\boldsymbol{r}_{m_2}} = \text{const}, \quad (3.5)$$

where the functional K_S is given by (3.3), and

$$E_r = \tfrac{1}{2}\int \left[(\nabla\psi_{1,r})^2 + \frac{\Lambda_1 s_1}{\Lambda_2 s_2}(\nabla\psi_{2,r})^2 + \Lambda_1\left(\psi_{1,r}^2 + \frac{s_1}{s_2}\psi_{2,r}^2 - 2s_1\psi_{1,r}\psi_{2,r}\right)\right]\mathrm{d}x\,\mathrm{d}y, \quad (3.6)$$

$$E_{sr} = \int\left(\psi_{1,s}\psi_{1,r} + \frac{\Lambda_1 s_1}{\Lambda_2 s_2}\psi_{2,s}\psi_{2,r}\right)\mathrm{d}x\,\mathrm{d}y + \tfrac{1}{2}\int\left(\psi_{1,s}^2 + \frac{\Lambda_1 s_1}{\Lambda_2 s_2}\psi_{2,s}^2\right)\mathrm{d}x\,\mathrm{d}y. \quad (3.7)$$

The positive-definite functional E_r coincides with the energy of the regular component. In this system, the last two terms on the left-hand side of (3.5) and the first term on the right-hand side of (3.7) represent the energy of interaction between the singular and the regular components. The function K_S and the second term on the right-hand side of (3.7) represent the self-interaction energy of the singular mode. If $p = 0$, i.e. if the singular vortices are conventional point vortices, integral (3.5) reduces to the sum of energy of the regular component, E_r, the energy of interaction of the point vortices, K_S, and the energy of interaction between the point vortices and the regular component (the last two terms in (3.5)).

Multiplication of the two sides of (2.22*a*) by $q_{1,r} + p^2\psi_{1,s}$ at $i = 1$ and by $(\Lambda_1 s_1/\Lambda_2 s_2)(q_{2,r} + p^2\psi_{2,s})$ at $i = 2$, integration over the (x, y)-plane, and summation of the results leads to an invariant that can be considered as an alternative form of the enstrophy conservation:

$$\tfrac{1}{2}\int\left[(q_{1,r} + p^2\psi_{1,s})^2 + \frac{\Lambda_1 s_1}{\Lambda_2 s_2}(q_{2,r} + p^2\psi_{2,s})^2\right]\mathrm{d}x\,\mathrm{d}y = -\beta\left(\sum_{m_1} A_{1,m_1} y_{m_1} + \frac{\Lambda_1 s_1}{\Lambda_2 s_2}\sum_{m_2} A_{2,m_2} y_{m_2}\right) = \text{const}. \quad (3.8)$$

In the absence of the singular component, integral (3.8), like (3.2), transforms into a conventional enstrophy invariant. Note that any two of the three integrals (3.2), (3.5), and (3.8) are independent, and the third integral is their linear combination.

3.3. *Mass and momentum integrals*

The set of invariants of the mixed singular–regular system (2.22) is completed by deriving integrals of mass and momentum conservation. First, the regular-component equation (2.22*a*) is integrated over the (x, y)-plane. According to the definition of a singular vortex

$$\int \psi_{i,s} \mathrm{d}x\, \mathrm{d}y = \text{const}, \quad i = 1, 2, \tag{3.9}$$

(see (2.21)). Therefore, from (2.22*a*) we obtain:

$$\int \psi_{i,r} \mathrm{d}x\, \mathrm{d}y = \text{const}, \quad i = 1, 2. \tag{3.10}$$

These two equalities represent the mass conservation by a mixed singular–regular system.

For this system, conservation of momentum along the x-axis is obtained by multiplying the two sides of (2.22*a*) by x at $i = 1$ and $(\Lambda_1 s_1/\Lambda_2 s_2)x$ at $i = 2$, integration over the (x, y)-plane, and summation of the results. Using (2.19) and (2.22*b*, *c*) after some algebra yields:

$$\begin{aligned} &\frac{\partial}{\partial t} \int x \left[(s_1 - 1)(\psi_{1,r} + \psi_{1,s}) + \frac{s_1}{s_2}(s_2 - 1)(\psi_{2,r} + \psi_{2,s}) \right] \mathrm{d}x\, \mathrm{d}y \\ &\quad = \frac{\beta}{\Lambda_1} \int \left[\psi_{1,r} + \psi_{1,s} + \frac{\Lambda_1 s_1}{\Lambda_2 s_2}(\psi_{2,r} + \psi_{2,s}) \right] \mathrm{d}x\, \mathrm{d}y = \text{const}. \end{aligned} \tag{3.11}$$

Constancy of the right-hand side in (3.11) is due to the mass conservation integrals (3.9) and (3.10). By a similar procedure, the momentum conservation along the y-axis is obtained:

$$\int y \left[(s_1 - 1)(\psi_{1,r} + \psi_{1,s}) + \frac{s_1}{s_2}(s_2 - 1)(\psi_{2,r} + \psi_{2,s}) \right] \mathrm{d}x\, \mathrm{d}y = \text{const}. \tag{3.12}$$

From (3.11) it is evident that the x-component of momentum linearly grows with time, on the β-plane, where as it is constant on the f-plane. The y-component remains constant on both the β- and f-planes, as seen in (3.12). Note that the momentum conservation laws (3.11) and (3.12) are non-trivial only in the models, where s_1 and s_2 are determined by formulae (2.2) and (2.4).

3.4. *Straightforward outcomes of the enstrophy and energy integrals on a β-plane*

3.4.1. *Single vortex*

Consider an individual singular vortex with its singular intrinsic vorticity confined to the upper layer, i.e. $N_1 = 1, A_{1,1} = A$, and $N_2 = 0$ in (2.21). The singular streamfunction then becomes:

$$\psi_{1,s} = A\psi_{1,s}^{u}(|\boldsymbol{r} - \boldsymbol{r}_0|), \quad \psi_{2,s} = A\psi_{2,s}^{u}(|\boldsymbol{r} - \boldsymbol{r}_0|), \tag{3.13}$$

where $\boldsymbol{r} = \boldsymbol{r}_0(t)$ is the trajectory of the vortex. Obviously, $K_S = 0$ in this case and, therefore, differentiation of (3.2) with respect to time yields:

$$\dot{S}_r - \beta A \dot{y}_0 = 0, \tag{3.14}$$

with a dot designating the time derivative.

Assume that, at an initial instant, the regular streamfunction, ψ_{ir}, is zero. On the β-plane, an individual singular vortex will not be a stationary state, and thus, necessarily, will start to generate a regular flow component. Therefore, at the initial stage of this generation, $\dot{S}_r > 0$ and, by virtue of (3.14), $A\dot{y}_0 > 0$. This implies that, at least at early times, a cyclonic singular vortex ($A > 0$) will move northward, and an anticyclonic vortex ($A < 0$) will move southward. Obviously, the same is valid for a vortex confined to the lower layer. Thus, qualitatively, the initial evolution of an individual monopole on the β-plane in two-layer fluid is the same as in the barotropic case (cf. Reznik 1992).

3.4.2. *Vortex pair*

Consider an ensemble composed of two singular vortices, one of which being confined to the upper layer and the other to the lower layer. In this case

$$\psi_{1,s} = A_1\psi^u_{1,s}(|\boldsymbol{r}-\boldsymbol{r}_1|) + A_2\psi^l_{1,s}(|\boldsymbol{r}-\boldsymbol{r}_2|), \tag{3.15a}$$

$$\psi_{2,s} = A_1\psi^u_{2,s}(|\boldsymbol{r}-\boldsymbol{r}_1|) + A_2\psi^l_{2,s}(|\boldsymbol{r}-\boldsymbol{r}_2|), \tag{3.15b}$$

and the function K_S becomes

$$K_S = CA_1A_2[K_0(p_-r_{1,2}) - K_0(p_+r_{1,2})], \tag{3.16}$$

where $r_{1,2} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$, and $C = \Lambda_1 s_1/(2\pi(\alpha^+ - \alpha^-)\Lambda_2 s_2) > 0$. By also assuming that, at $t = 0$, the locations of the two vortices coincide, i.e. $r_{1,2} = 0$, and the regular-flow component is absent, we obtain:

$$\psi_1 = \psi_{1,s} = \frac{\alpha^-}{2\pi(\alpha^+ - \alpha^-)}(A_1 + \alpha^+A_2)K_0(p_-r) - \frac{\alpha^+}{2\pi(\alpha^+ - \alpha^-)}(A_1 + \alpha^-A_2)K_0(p_+r), \tag{3.17a}$$

$$\psi_2 = \psi_{2,s} = -\frac{1}{2\pi(\alpha^+ - \alpha^-)}(A_1+\alpha^+A_2)K_0(p_-r) + \frac{1}{2\pi(\alpha^+ - \alpha^-)}(A_1 + \alpha^-A_2)K_0(p_+r). \tag{3.17b}$$

On the β-plane, the vortex pair under consideration can be shown to be non-stationary. Therefore, a regular component arises and the vortices start separating, i.e. $\dot{S}_r > 0$ and $\dot{r}_{1,2} > 0$ at early times. Differentiation of (3.16) yields:

$$\dot{K}_S = -CA_1A_2\dot{r}_{1,2}[p_-K_1(p_-r_{1,2}) - p_+K_1(p_+r_{1,2})]. \tag{3.18}$$

According to (2.12) and (2.19*c*), $p_- < p_+$. Therefore, for small values of $r_{1,2}$, the following inequality holds:

$$p_-K_1(p_-r_{1,2}) - p_+K_1(p_+r_{12}) > 0, \quad r_{1,2} \ll 1. \tag{3.19}$$

If the vortices of this pair are of different signs, i.e. if $A_1A_2 < 0$, then according to (3.18), $\dot{K}_S > 0$. Hence, (3.2) in this case implies:

$$A_1\left(\dot{y}_1 - \frac{\Lambda_1 s_1}{\Lambda_2 s_2}\left|\frac{A_2}{A_1}\right|\dot{y}_2\right) > 0. \tag{3.20}$$

When amplitudes A_1 and A_2 satisfy the additional condition of

$$\frac{\Lambda_1 s_1}{\Lambda_2 s_2}\left|\frac{A_2}{A_1}\right| = 1, \tag{3.21}$$

it follows from (3.20) that

$$A_1\dot{y}_{12} > 0. \tag{3.22}$$

Thus, under condition (3.21), the cyclonic vortex starts moving northward, and the anticyclonic vortex southward, while the rise of a separation between the cyclone and anticyclone induces the tendency of the vortex pair to propagate eastward (at least at early times).

Condition (3.21) has a clear physical meaning in the rigid-lid model (2.5). In this case, $\alpha^- = -1$ and $\alpha^+ = H_2/H_1$, so the two vertical normal modes (2.8) are just the barotropic and baroclinic modes. In these circumstances, under condition (3.21), equations (3.17) reduce to

$$\psi_1|_{t=0} = -\frac{H_2}{H_1}\frac{A}{2\pi}K_0(p_+r), \quad \psi_2|_{t=0} = \frac{A}{2\pi}K_0(p_+r), \quad A = A_1 - A_2, \tag{3.23}$$

which imply that the initial state is a purely baroclinic vortex.

In fact, the conclusion that the development of a regular component tilts the axis of an initially baroclinic vortex pair and induces its eastward drift, generalizes to singular vortices the result obtained previously with point vortices (Reznik, Grimshaw & Sriskandarajah 1997). Long-term numerical simulations with distributed fully baroclinic coaxial vortical pairs set as initial states (Kizner, Berson & Khvoles 2002) demonstrated that, as time passes, the growth of separation between the upper and lower vortices ceases, and the vortex pair gradually transforms into an eastward-travelling baroclinic modon.

4. Stability of a point-vortex pair on the *f*-plane

4.1. *Statement of the problem and result*

The enstrophy and energy invariants (3.2), (3.5) allow examination of nonlinear stability of an arbitrary *f*-plane point-vortex pair in the models given by (2.2), (2.4) or (2.6). When the regular component is absent, the dynamics of a vortex pair are simple (Gryanik 1983). Namely, the pair, which is comprised of vortices with amplitudes A_1 and A_2, executes either a uniform circular motion around a fixed point (centre of mass) if $A_1 \neq -A_2$, or a uniform-speed motion if $A_1 = -A_2$. The separation between the vortices, $r_{1,2}$, is a constant of motion. Given the amplitudes A_1 and A_2, the translation speed (at $A_1 = -A_2$), or the centre location and the angular velocity (at $A_1 \neq -A_2$) are determined by the separation $r_{1,2}$ only.

The stability of an ensemble of discrete vortices can be determined in several ways. According to the conventional definition, which does not take into account any regular flow, an ensemble is regarded as stable if, in response to small initial perturbations of the coordinates of the vortices, changes in distances between each two vortices remain small. Thus, a stable ensemble is allowed to move as a whole, but a perturbation in the mutual arrangement of the vortices should not grow. According to this definition, in the absence of a regular flow, the motion of an *f*-plane point-vortex pair is stable: small changes in $r_{1,2}$ result in small changes of the centre of mass (at $A_1 \neq -A_2$) and the speed of the pair only.

In principle, a regular perturbation imposed on the vortex pair can result in significant changes in the separation between the vortices and in their velocities. This requires a generalization of the stability definition for mixed discrete–regular systems by incorporation of a regular component in the initial perturbation. As shown below, if at the initial moment the perturbation is small, and the enstrophy and energy of the regular component of the perturbation are finite, then (i) the perturbation remains small at all subsequent times, and (ii) it causes only small changes in the separation between the vortices. This means that any point-vortex pair, either rotating or translating, is stable in the broader sense, i.e. relative to small perturbations that contain regular components. Up to now, the stability of point-vortex ensembles seems to have been investigated only with respect to initial perturbations of the coordinates of the vortices, without taking into account any regular component.

4.2. *Outline of analysis*

On the f-plane $p = 0$, i.e. singular vortices are just conventional point vortices with the streamfunctions (2.10), (2.11) (see §§ 2.2 and 2.3). For a point-vortex ensemble, the first two conservation laws (3.2), (3.5) become:

$$S_r = \tfrac{1}{2}\int\left(q_{1,r}^2 + \frac{\Lambda_1 s_1}{\Lambda_2 s_2} q_{2,r}^2\right)\mathrm{d}x\,\mathrm{d}y = \Omega = \text{const}, \tag{4.1}$$

$$E_r + K_S - \sum_{m_1} A_{1,m_1}\,\psi_{1,r}|_{\boldsymbol{r}=\boldsymbol{r}_{m_1}} - \frac{\Lambda_1 s_1}{\Lambda_2 s_2}\sum_{m_2} A_{2,m_2}\,\psi_{2,r}|_{\boldsymbol{r}=\boldsymbol{r}_{m_2}} = E_0 = \text{const}. \tag{4.2}$$

Relationship (4.1) means that the enstrophy of the regular component is conserved. In the absence of the regular component, according to (4.2) the energy of interaction between vortices, K_S, is conserved.

Consider a point-vortex ensemble perturbed by some regular field, and Ω to be the enstrophy of the regular component of the perturbed system. Using enstrophy conservation (4.1) and equations (2.23), which relate the intrinsic regular vorticity with the regular streamfunctions, we estimate in terms of Ω the absolute value of the regular streamfunction and the energy of the regular component, E_r, in layer i. These estimates are merely the inequalities $|\psi_{i,r}| < C_i\sqrt{\Omega}$ and $E_r < \hat{E}_r\Omega$, where C_i and $\hat{E}_r$ are constants that depend on the model parameters only (such as geometry and stratification). If the enstrophy, Ω, is sufficiently small, then E_r and the energy of interaction between the regular and singular components (the last two terms on the left-hand side of (4.2)) are small compared to the total energy E_0. This means that a sufficiently small regular perturbation can cause only small changes in K_S. In the case of a vortex pair, K_S is a continuous function of only one argument, the separation $r_{1,2}$ between the two vortices. Therefore, small changes in K_S are possible only if changes in $r_{1,2}$ are small, i.e. only when the vortex pair is stable.

4.3. *Auxiliary estimates*

In order to estimate the moduli $|\psi_{i,r}|$, the definition of normal modes (2.8) is applied to the regular components of the streamfunctions and intrinsic vorticities:

$$\psi_{1,r} = \frac{\alpha^+\psi_r^- - \alpha^-\psi_r^+}{\alpha^+ - \alpha^-}, \quad \psi_{2,r} = \frac{\psi_r^- - \psi_r^+}{\alpha^+ - \alpha^-}, \tag{4.3a}$$

$$q_{1,r} = \frac{\alpha^+ q_r^- - \alpha^- q_r^+}{\alpha^+ - \alpha^-}, \quad q_{2,r} = \frac{q_r^- - q_r^+}{\alpha^+ - \alpha^-}. \tag{4.3b}$$

By inserting expressions (4.3) into (2.23) and resolving the resulting equations with respect to $\psi_r^\pm$, we obtain:

$$\psi_r^\pm = -\frac{1}{2\pi}\int q_r^\pm(\boldsymbol{r}', t)K_0(p_\mp\,|\boldsymbol{r}-\boldsymbol{r}'|)\,\mathrm{d}x'\,\mathrm{d}y', \tag{4.4}$$

where the coefficient $p_\mp$ at $|\boldsymbol{r}-\boldsymbol{r}'|$ is given by (2.12). Application of the Cauchy–Schwartz–Bunyakowsky inequality to the right-hand side of (4.4) yields the following estimate:

$$|\psi_r^\pm| \leqslant C_\mp \sqrt{\int (q_r^\pm)^2 \,\mathrm{d}x\,\mathrm{d}y}, \quad C_\pm = \frac{1}{2\pi}\sqrt{\int K_0^2(p_\pm r)\,\mathrm{d}x\,\mathrm{d}y}. \tag{4.5}$$

By combining (4.3*a*, *b*) and (4.5), the sought-after estimates are obtained:

$$|\psi_{i,r}| \leqslant C_i\sqrt{\Omega}, \quad i=1,2, \tag{4.6}$$

where

$$C_1 = \frac{\alpha^+ C_1^- - \alpha^- C_1^+}{\alpha^+ - \alpha^-}, \quad C_2 = \frac{C_1^- + C_1^+}{\alpha^+ - \alpha^-}, \tag{4.7a}$$

$$C_1^\pm = C_\mp \sqrt{2(1+|\alpha^\pm|)\left(1+\frac{\Lambda_1 s_1}{\Lambda_2 s_2}|\alpha^\pm|\right)}. \tag{4.7b}$$

The constants C_1 and C_2 given by (4.7*a*) are positive since, by virtue of (2.9), $\alpha^- < 0$ and $\alpha^+ > 0$.

In estimation of energy of the regular flow, the following four inequalities derived from (4.1) and (2.23) are useful:

$$\int q_{1,r}^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant 2\Omega, \quad \int q_{2,r}^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant 2\Omega\frac{\Lambda_2 s_2}{\Lambda_1 s_1}, \tag{4.8a}$$

$$\int \psi_{1,r}^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{\Lambda_1 s_1 s_2 + \Lambda_2}{\Lambda_1\Lambda_2(1-s_1s_2)^2}\frac{2\Omega}{\Lambda_1}, \quad \int \psi_{2,r}^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{s_2}{s_1}\frac{\Lambda_1 + \Lambda_2 s_1 s_2}{\Lambda_1\Lambda_2(1-s_1s_2)^2}\frac{2\Omega}{\Lambda_1}. \tag{4.8b}$$

The energy of the regular component, E_r can be represented as

$$E_r = -\tfrac{1}{2}\int\left(\psi_{1,r}q_{1,r} + \frac{\Lambda_1 s_1}{\Lambda_2 s_2}\psi_{2,r}q_{2,r}\right), \tag{4.9}$$

which allows the estimate:

$$E_r \leqslant \tfrac{1}{2}\int\left(|\psi_{1,r}q_{1,r}| + \frac{\Lambda_1 s_1}{\Lambda_2 s_2}|\psi_{2,r}q_{2,r}|\right)\mathrm{d}x\,\mathrm{d}y. \tag{4.10}$$

Application again of the Cauchy–Schwartz–Bunyakowsky inequality together with estimates (4.8) and (4.6), yields the following estimate for E_r:

$$E_r \leqslant \hat{E}_r\Omega, \quad \hat{E}_r = \frac{1}{\sqrt{\Lambda_1\Lambda_2}(1-s_1s_2)}\left(\sqrt{\frac{\Lambda_2}{\Lambda_1}+s_1s_2} + \sqrt{\frac{\Lambda_1}{\Lambda_2}+s_1s_2}\right). \tag{4.11}$$

4.4. *Stability of vortex pairs*

First, consider a pair of vortices positioned in the same (for example, upper) layer. In this case, the general expression (3.3) for the energy of interaction between the vortices, K_S, becomes

$$K_S = A_1A_2G_1(r_{1,2}), \quad G_1 = \frac{1}{4\pi}[\gamma^+K_0(p_+r_{1,2}) + \gamma^-K_0(p_-r_{1,2})], \tag{4.12}$$

where $\gamma^+ = \alpha^+/(\alpha^+ - \alpha^-)$, $\gamma^- = -\alpha^-/(\alpha^+ - \alpha^-)$, and $\gamma^\pm > 0$. We assume that $r_{1,2} = r_{1,2}^{(0)} > 0$ at $t = 0$,

From the energy conservation (4.2), it follows that, at any moment $t > 0$,

$$\begin{aligned} \left|K_S - K_S^{(0)}\right| \leqslant \left|E_r\right| + \left|E_r^{(0)}\right| + \left|A_1\right|\left(|\psi_{1,r}|_{r=r_1}| + |\psi_{1,r}^{(0)}|_{r=r_1^{(0)}}|\right) \\ + \left|A_2\right|\left(|\psi_{1,r}|_{r=r_2}| + |\psi_{1,r}^{(0)}|_{r=r_2^{(0)}}|\right), \end{aligned} \tag{4.13}$$

where superscript (0) denotes the quantities at $t = 0$. Relationships (4.12) and (4.13) along with inequalities (4.6) and (4.11) result in the following estimate of the range of variability of the function $G_1(r_{1,2})$:

$$\left|G_1(r_{1,2}) - G_1\left(r_{1,2}^{(0)}\right)\right| \leqslant \frac{2C_1}{|A_1A_2|}\left[|A_1| + |A_2| + \frac{\hat{E}_r}{C_1}\sqrt{\Omega}\right]\sqrt{\Omega} = R_1. \tag{4.14}$$

In (4.14), the regular enstrophy Ω is determined by the initial conditions, the coefficients $\hat{E}_r$ and C_1 are determined by the model parameters (see (4.7) and (4.11)), and the amplitudes A_1 and A_2 are fixed. In view of (4.12), the function $G_1(z)$ is monotonic. Therefore, for sufficiently small initial regular fields, i.e. when the regular enstrophy Ω and the parameter R_1 are small, according to (4.14) the quantity $|r_{1,2} - r_{1,2}^{(0)}|$ is also small:

$$\left|r_{1,2} - r_{1,2}^{(0)}\right| = O(R_1). \tag{4.15}$$

More specifically, for small R_1, the range of $r_{1,2}$ can be estimated as

$$\left|r_{1,2} - r_{1,2}^{(0)}\right| \leqslant \frac{2C_1}{\left|G_1'\left(r_{1,2}^{(0)}\right)A_1A_2\right|}\left[|A_1| + |A_2| + \frac{\hat{E}_r}{C_1}\sqrt{\Omega}\right]\sqrt{\Omega}. \tag{4.16}$$

When vortices positioned in different layers are considered, (3.3) yields:

$$K_S = A_1A_2G_2(r_{1,2}), \quad G_2 = \frac{1}{2\pi(\alpha^+ - \alpha^-)}\frac{A_1s_1}{A_2s_2}[K_0(p_-r_{1,2}) - K_0(p_+r_{1,2})]. \tag{4.17}$$

Following the line of reasoning used in the previous case, we arrive at the inequality:

$$\left|G_2(r_{1,2}) - G_2\left(r_{1,2}^{(0)}\right)\right| \leqslant \frac{2C_1}{\left|A_1A_2\right|}\left[|A_1| + \frac{A_1s_1}{A_2s_2}\frac{C_2}{C_1}|A_2| + \frac{\hat{E}_r}{C_1}\sqrt{\Omega}\right]\sqrt{\Omega}. \tag{4.18}$$

The function $G_2(z)$ is monotonic; therefore, for a sufficiently small initial regular perturbation, the range of $r_{1,2}$ is estimated as

$$\left|r_{1,2} - r_{1,2}^{(0)}\right| \leqslant \frac{2C_1}{\left|G_2'\left(r_{1,2}^{(0)}\right)A_1A_2\right|}\left[|A_1| + \frac{A_1s_1}{A_2s_2}\frac{C_2}{C_1}|A_2| + \frac{\hat{E}_r}{C_1}\sqrt{\Omega}\right]\sqrt{\Omega}. \tag{4.19}$$

According to the estimates (4.16) and (4.19), at $\beta = 0$, any pair of point vortices in the two-layer models (2.2) and (2.4) is nonlinearly stable relative to an arbitrary regular perturbation with a sufficiently small enstrophy. This implies stability to any combined initial perturbation consisting of a small regular component and a small change in the coordinates of the vortices.

The result regarding the stability of a vortex pair can easily be extended to the $1\frac{1}{2}$-layer model (2.6). However, as seen from (4.8*b*), (4.11), this analysis fails for the two-layer model (2.5) with a rigid-lid condition, i.e. when $s_1 = s_2 = 1$. Mathematically, this is because we are currently unable to estimate the regular energy E_r in terms of the regular enstrophy Ω. Once this technical problem is overcome, the proof of stability of two-layer vortical pairs under the rigid-lid condition should be feasible.

5. Summary

We developed a theory of quasi-geostrophic singular vortices that occupies an intermediate position between discrete and fully continuous multilayer models. On the f-plane, within the framework of this theory, singular vortices are the conventional point vortices, whose potential vorticity is concentrated in a point, being proportional to Dirac's delta-function. In the presence of the β-effect, more general singular vortices arise that differ from point vortices. The intrinsic vorticity of such a singular vortex, along with the delta-function term, contains an additional term that decays exponentially far enough from the centre of the vortex. This form of a singular vortex on the β-plane was chosen because any β-plane modon, i.e. a distributed localized steadily translating structure, can be represented as a continuous superposition of singular vortices of this kind.

In addition to the flow associated with the singular vortices themselves, a regular flow can exist. This paper focused on the interaction of singular vortices with each other and also with the regular flow. We derived equations that describe the cooperative evolution of the vortices and the regular field (§ 2), and obtained the integrals of conservation of enstrophy, energy, momentum and mass in such a vortical system (§ 3).

Analysis of the enstrophy integral at $\beta \neq 0$ permitted investigation of the initial stage of evolution of an individual singular vortex confined to one of the layers. The regular flow generated by the vortex transports the vortex meridionally: a cyclone starts travelling northward, and an anticyclone southward. This evolution is similar to that which occurs with barotropic vortices. Another direct consequence of the enstrophy conservation is related to the initial stage of evolution of a purely baroclinic β-plane configuration comprised of a pair of singular vortices that have the same coordinates and are of opposite signs. The initially vertical axis of such a baroclinic vortex pair must tilt, so the cyclone shifts to the north, and the anticyclone to the south. This shifting causes a tendency of the vortex pair to propagate eastward.

The analysis of stability of two-layer vortex pairs herein was extremely challenging. So far, the stability of point-vortex ensembles has been examined only with respect to perturbations in the coordinates of the vortices. The effect of a regular flow, on a point-vortex pair strongly depends on whether or not the energy and enstrophy of this regular flow are finite. For example, a point-vortex dipole embedded in a rectilinear shear flow with whatever weak constant intrinsic vorticity in each layer will always disintegrate (Gryanik 1983). In this example, the energy and enstrophy of the background flow are infinite. We proved the nonlinear stability of an arbitrary point-vortex pair on the f-plane with respect to any small regular perturbation with finite energy and enstrophy. Our proof is based on the energy and enstrophy invariants, and finiteness of the regular energy and enstrophy is critical for this stability.

G.R. gratefully acknowledges the hospitality extended to him by Bar-Ilan University during his stay in Israel, and the support from Russian Foundation for Basic Research, Grant 05-05-64212. Z. K. acknowledges the support from Israel Science Foundation, 628/06. The authors thank the Centre for Academic and Educational Relations of the Humanities, Hebrew University, for partial support of this research and V. Gryanik and V. Zeitlin for helpful discussions.

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